Hydrodynamic Stability of Viscoelastic Fluids: Importance of Fluid Model, Overstability, and Form of Disturbance

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A linearized stability analysis has been applied to a fluid flowing in a gravity field between horizontal planes in Couette flow under conditions such that the temperature of the bottom plane exceeds that of the top. It is shown that, under conditions likely to be encountered with polymer solutions, oscillatory instabilities will not be controlling. Criteria are offered for ascertaining when an analysis based upon a second-order fluid model may be expected to yield physically meaningful results. It is also shown that, for the fluid model considered, critical conditions for stability are not changed when disturbances which vary in the flow direction are substituted for those which are a function of the coordinate transverse to the flow.

Engineers and scientists concerned with research, development, and design of systems associated with polymer melts and polymer solutions are well aware of the difficulties posed by the complex rheological behavior of these fluids. In particular, measurement of nonisotropic normal stresses which are generated in shearing flows is acknowledged to be as difficult as it is important for fluid characterization. Of the three material functions known to be necessary for characterization of viscometric flows, the shear dependent viscosity and the primary normal stress difference N_1 can be routinely measured, if one assumes that the secondary normal stress difference N_2 is negligible. However, though N_2 may be expected to be numerically smaller than N_1, N_2 appears to have considerable importance as a parameter for fluid characterization. This is especially true, as discussed in some detail below, for description of certain hydrodynamic stability problems. Measurement of N_2 has proved extremely difficult.

Ginn and Denn (*1*) have suggested that N_2 might be conveniently measured by indirect means. Denn and Roisman (*2*) have shown through an interesting series of experiments and accompanying analysis that the stability characteristics of a fluid in flow between coaxial cylinders is highly sensitive to the sign and magnitude of N_2, and that in addition to being an interesting stability problem in its own right, stability characteristics of coaxial cylinder flow can afford a sensitive means for measurement of N_2. This possibility is implicit also in the work of others, notably Giesekus (*3*) and Goddard and Miller (*4*). The latter authors have indicated the problems which can arise when one uses viscometric functions to describe inherently nonviscometric flows.

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In a recent paper (*5*) we have shown that N_2 is a sensitive parameter governing the stability of a fluid which is contained between horizontal parallel plates and is subjected to superposition of a temperature gradient (the Bénard or Rayleigh-Jeffreys problem) upon plane Couette flow, as shown in Figure 1. The configuration of Figure 1 can be considered an idealization of processing situations in which a viscoelastic fluid is subjected to spatial temperature variation while being sheared in a gravitational field. A rather complete discussion of this problem and its relation to classical problems in hydrodynamic stability is included in the earlier paper. In this paper we show that the restricted range of applicability of the previous analysis can be significantly broadened without any essential change in the conclusions.

A second important aspect of this paper concerns the use of viscoelastic fluid models in hydrodynamic stability analysis. The second-order fluid model is easily applied to

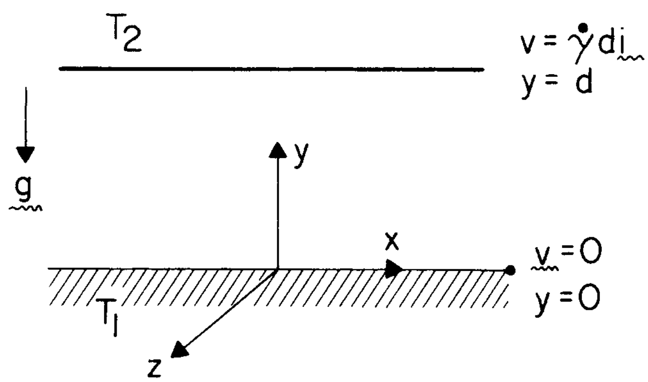

Fig. 1. Boundary conditions for plane Couette flow with superposed temperature gradient.

stability problems but, as we shall show later, it can lead to physically meaningless results under certain conditions. On the other hand, integral models, though difficult to use in most analyses, provide more realistic results. It is shown that for plane Couette flow with superposed temperature gradient there are regions where both models give the same results, and thus the second-order fluid model may be used. Behavior of the exponential growth rate near the neutral disturbance is of critical importance in determining validity of the results obtained with a second-order fluid model.

BACKGROUND

Constitutive Equations

In the paper cited (*5*), an analysis was carried out for two different forms of constitutive equations. These were Model A, a "generalized" second-order model,

$$\mathbf{S} + p\mathbf{I} = \alpha_0 \mathbf{A}_1 + \alpha_1 \mathbf{A}_1^2 + \alpha_2 \mathbf{A}_2 \tag{1}$$

and Model B, an integral expression similar in form to the network rupture model employed by Tanner and Simmons (*6*) and Tanner (*7*).

$$\mathbf{S} + p\mathbf{I} = \int_{t-\tau_R}^{t} m[(t-t'), \mathrm{II}(t')] \left[\left(1+\frac{\delta}{2}\right)\mathbf{B} + \frac{\delta}{2}\mathbf{C}\right] dt' \tag{2}$$

Model *B* is also closely related to the constitutive equation of Bird and Carreau (*8*), who have discussed its molecular significance.

It was found in (*5*) that both Models A and B led to the same set of differential equations which were to be satisfied at conditions of nonoscillatory marginal stability. In contrast to the case for Newtonian fluids there is coupling between the equations describing momentum and energy disturbances. This fact leads, for viscoelastic fluids, to a critical Rayleigh number (N_{Ra_c}) which is highly sensitive to the sign and magnitude of the second normal stress difference, thus offering the possibility that the flow may be appropriate for measurement of N_2 by indirect means.

Exchange of Stabilities

Linearized stability analysis is characterized by study of the behavior with time of a small disturbance of the form $A(\mathbf{x})e^{\sigma t}$ imposed upon a basic solution to the time-independent equations of mass, momentum, and energy transport. In many cases one seeks those conditions for which $\sigma = 0$, indicating a state of marginal stability at which disturbances neither grow nor decay. This approach is incomplete, however, since in general σ is complex, and we have $\sigma = \sigma_r + i\sigma_i$. Thus one can imagine an indefinite number of states of marginal stability ($\sigma_r = 0$), each corresponding to a different σ_i. If the onset of instability is characterized by a disturbance which is not oscillatory in time ($\sigma_i = 0$), then the state of marginal stability can be found by setting $\sigma = 0$. When such a procedure is justified, the principle of *exchange of stabilities* (*9*) is said to be operative. However, if the state of marginal stability is for a disturbance with an amplitude that is oscillatory in time, then $\sigma_i \neq 0$ and the marginally stable state is described as one of overstability. One is of course most interested in the condition which bounds the region of flow and material parameters for which $\sigma_r < 0$ for *any* form of small disturbance, this condition being known as the critical state. For many Newtonian problems one can prove that the principle of exchange of stabilities is valid at the critical state, and hence the case $\sigma_i \neq 0$ can be ignored. Such proofs, however, do not normally carry over to analogous non-Newtonian problems. Nevertheless, because of their complexity, almost all previous stability analyses of non-Newtonian fluids, including our own work, have been restricted to marginal states where $\sigma = \sigma_r = \sigma_i = 0$. Vest and Arpaci (*10*) and Green (*11*) working on problems related to the present one have indicated that for real fluids the stability of most systems seems to be controlled by a nonoscillatory mode of disturbance. The analysis presented below indicates that, for both fluid models, nonoscillatory states of marginal stability are controlling, and hence one is justified in setting $\sigma = 0$ when looking for the critical states of stability. Furthermore, if it is permissible to set $\sigma = 0$, then, as noted earlier, the solution based upon Model A is also applicable for Model B.

Form of Disturbance

Previous work dealing with the present and related problems has been confined to disturbances which are dependent upon only two dimensions. It is quite satisfactory to do this with Newtonian fluids since one can apply Squire's theorem, which shows that two-dimensional disturbances control the state of critical stability. Unfortunately, as pointed out by Lockett (*12*), an equally general equivalent of Squire's theorem cannot be applied to non-Newtonian systems.

Prior work (*13, 14*) with the combined Couette-Bénard problem for Newtonian fluids has produced an interesting result: If the disturbance is independent of the flow direction, the Couette and Bénard portions of the flow remain uncoupled, and the stability is governed by a critical Rayleigh number which is unaltered regardless of the shear rate ($N_{Ra_c} = 1{,}708$ for fixed boundaries at constant temperature). However, if the disturbance is a function of the coordinate in the flow direction but is independent of the transverse direction (z, Figure 1) then the two phenomena are coupled and N_{Ra_m} (marginal state for a particular disturbance) is a function of shear rate. However, the lowest value of N_{Ra_m} occurs in the former case, so that N_{Ra_c} is unaffected by the presence of shear.

In our previous paper (*5*) we were restricted to disturbances of the form $A(z, y)e^{\sigma t}$. Since we obtained a strong coupling between the heat and momentum transfer portions of the problem, we were anxious to look at disturbances of the form $A(x, y)e^{\sigma t}$, anticipating that an even stronger coupling might be present in the non-Newtonian case.

Scope of Present Work

The purpose of the present paper is to offer a more complete analysis of the problem than was available heretofore. In particular we question the following assumptions and approximations which were made at the time of our earlier analysis:

1. Since the principle of exchange of stabilities cannot be proved for the system of interest, it is important to consider the possibility of oscillatory instabilities. Will the critical conditions for stability be altered by admission of a disturbance which grows or decays in an oscillatory manner?
2. States of marginal stability ($\sigma_r = \sigma_i = 0$) were found by an approximate solution of the set of relevant ordinary differential equations, employing Galerkin's method. Are the results seriously different from an exact solution to the differential equation?
3. The previous analysis considered disturbances to the flow which are functions of the y and z directions only (see Figure 1). Will the results be affected if one considers disturbances which oscillate along the flow direction?

Under conditions most likely to be of interest for real materials, all three of the approximations questioned are shown to be valid when applied to Model A. Since non-oscillatory disturbances are found to be controlling for both models, approximations 1 and 2 are also valid when applied to Model B. Finally, we present results for both models over a somewhat wider range of parameters than was previously possible, and we discuss the utility of a second order fluid model for stability analysis.

ANALYSIS

Returning to the question of overstability, we note that Davis (*15*) using ideas similar to those of Finlayson (*16, 17*), has shown how the principle of exchange of stabilities is related to the self-adjointness of the differential operators involved. Treating the nonselfadjoint part of the system as a small perturbation and assuming a regular perturbation expansion, convergence is proved and an estimate of the radius of convergence is obtained. Within the radius of convergence it is proved that a class of real, bounded perturbations of a real selfadjoint system preserves the reality of σ (that is, exchange of stabilities is valid). In general this method gives very conservative results in the sense that the asymmetry must be extremely small in order for convergence to be proved. As the asymmetry in the present problem may be of the same order of magnitude as the basic selfadjoint system, the theorem of Davis is not helpful to us, and the degree of importance of overstability is found by direct calculation.

Though a full three-dimensional treatment of the disturbance would be desirable in the present case, we have avoided the accompanying complexity by following the approach of Gallegher and Mercer (*13*) and of Deardorff (*14*) for Newtonian fluids. Thus we consider two sets of two-dimensional disturbances:

Case 1. Disturbances are spatially dependent only upon the y and z coordinates. This choice of coordinates permits correspondence with the results of Giesekus (*3*) and Goddard and Miller (*4*), who considered only axisymmetric disturbances in cylindrical Couette flow and then treated plane Couette flow as a limiting case when the radius of curvature approaches infinity.

Case 2. Disturbances are spatially dependent only upon the y and x coordinates (*13, 14*). The role of disturbances which are functions of the streamwise coordinate has only been considered for Model A.

Case 1. Overstability

Differential Equations for Model A

Linearized expressions for the components of the stress disturbance are readily found for Model A.

$$\begin{aligned}
S'_{xx} + p' &= 2\alpha_1 v'_{x,y} \\
S'_{yy} + p' &= 2\alpha_0 v'_{y,y} + 2(\alpha_1 + 2\alpha_2)\,\dot\gamma\, v'_{x,y} + 2\alpha_2 v'_{y,yt} \\
S'_{zz} + p' &= 2\alpha_0 v'_{z,z} + 2\alpha_2 v'_{z,zt} \\
S'_{xy} = S'_{yx} &= \alpha_0 v'_{x,y} + (2\alpha_1 + \alpha_2)\,\dot\gamma\, v'_{y,y} + \alpha_2 v'_{x,yt} \\
S'_{xz} = S'_{zx} &= \alpha_0 v'_{x,z} + (\alpha_1 + \alpha_2)\,\dot\gamma\, v'_{y,z} + \alpha_1\,\dot\gamma\, v'_{z,y} + \alpha_2 v'_{x,zt} \\
S'_{yz} = S'_{zy} &= \alpha_0 (v'_{y,z} + v'_{z,y}) + (\alpha_1 + 2\alpha_2)\,\dot\gamma\, v'_{x,z} + \alpha_2 (v'_{z,yt} + v'_{y,zt})
\end{aligned} \tag{3}$$

where a comma indicates partial differentiation. The momentum and energy balance equations reduce for Case 1 to

$$\begin{aligned}
\frac{\partial v'_x}{\partial \bar t} + v'_y \frac{\dot\gamma\, d}{U} &= \frac{1}{\rho_1 U}\left[\frac{\partial \tau'_{yx}}{\partial \xi} + \frac{\partial \tau'_{zx}}{\partial \eta}\right] \\
\frac{\partial v'_y}{\partial \bar t} &= \frac{\alpha d}{U} g T' - \frac{1}{\rho_1 U}\frac{\partial p'}{\partial \xi} + \frac{1}{\rho_1 U}\left[\frac{\partial \tau'_{yy}}{\partial \xi} + \frac{\partial \tau'_{zy}}{\partial \eta}\right] \\
\frac{\partial v'_z}{\partial \bar t} &= -\frac{1}{\rho_1 U}\frac{\partial p'}{\partial \eta} + \frac{1}{\rho_1 U}\left[\frac{\partial \tau'_{yz}}{\partial \xi} + \frac{\partial \tau'_{zz}}{\partial \eta}\right] \\
\frac{\partial T'}{\partial \bar t} - \frac{v'_y}{U}\beta d &= \frac{\kappa}{Ud}\nabla^2 T'
\end{aligned} \tag{4}$$

where $\xi = y/d - 1/2$

$\eta = z/d$

$U = \dot\gamma\, d$

$\bar t = (U/d)t$

$\boldsymbol{\tau} = \mathbf{S} + p\mathbf{I}$

The disturbances are decomposed into normal modes (*9*)

$$\begin{aligned}
\frac{v'_x}{U} &= -\epsilon X(\xi)\sin(\epsilon\eta)e^{\sigma\bar t} \\
\frac{v'_y}{U} &= \epsilon\Psi(\xi)\sin(\epsilon\eta)e^{\sigma\bar t} \\
\frac{v'_z}{U} &= \Psi_{,\xi}\cos(\epsilon\eta)e^{\sigma\bar t} \\
\frac{T'}{(T_1 - T_2)} &= \epsilon T(\xi)\sin(\epsilon\eta)e^{\sigma\bar t} \\
\frac{p'}{\rho_1 U^2} &= \epsilon P(\xi)\sin(\epsilon\eta)e^{\sigma\bar t}
\end{aligned} \tag{5}$$

where ϵ = dimensionless wave number

$\sigma = \sigma_r + i\sigma_i$ = complex growth rate

Combining (3), (4), and (5), one finally obtains

$$\begin{aligned}
(D^2 - \epsilon^2 - N_{Pe}\sigma)\Theta &= -\epsilon^2 N_{Ra}\Psi \\
\Theta &= -N_{Re}\sigma(D^2 - \epsilon^2)\Psi + (D^2 - \epsilon^2)^2\Psi + G\epsilon^2(D^2 - \epsilon^2)X + J\sigma(D^2 - \epsilon^2)^2\Psi \\
(D^2 - \epsilon^2)X + J\sigma(D^2 - \epsilon^2)X - N_{Re}\sigma X &= (F(D^2 - \epsilon^2) - N_{Re})\Psi
\end{aligned} \tag{6}$$

where $\Theta = \dfrac{\rho_1 \alpha d^2 g (T_1 - T_2)\epsilon^2}{U\alpha_0} T(\xi)$

$$N_{Re} = \frac{\rho_1 d^2 \dot\gamma}{\alpha_0}$$

$$N_{Ra} = \frac{\rho_1 d^4 g \alpha \beta}{\kappa \alpha_0}$$

$$N_{Pe} = N_{Re}\, N_{Pr} = \frac{d^2 \dot\gamma}{\kappa}$$

$$F = \frac{(\alpha_1 + \alpha_2)}{\alpha_0}\dot\gamma$$

$$G = \frac{(\alpha_1 + 2\alpha_2)\,\dot\gamma}{\alpha_0}$$

$$J = \frac{\alpha_2\,\dot\gamma}{\alpha_0}$$

$$D = d/d\xi$$

Equations (6) are subject to the same restrictions on fluid properties as those cited in (5, 9). Essentially, we are requiring that the Boussinesq assumption, which is usually invoked in problems of convective instability, is valid. Use of Equation (1) of course implies that it is a satisfactory constitutive equation not only for the viscometric steady flow but also for small departures from it (3). In addition we have assumed in Equations (6) that the flow disturbance induces perturbations in the α_i which are of smaller order than those in A_i. This assumption has been discussed previously (5).

A state of neutral stability is of course characterized by setting $\sigma_r = 0$ in Equations (6). In the special case where exchange of stabilities is valid, Equations (6) reduce to those solved approximately in (5).

$$\begin{aligned} \Theta &= (D^2 - \epsilon^2)^2\Psi + GF\epsilon^2(D^2 - \epsilon^2)\Psi - GN_{Re}\epsilon^2\Psi \\ (D^2 - \epsilon^2)\Theta &= -\epsilon^2 N_{Ra}\Psi \end{aligned} \tag{7}$$

For the situation shown in Figure 1 (the case of fixed boundaries at constant temperature), boundary conditions are

$$\Theta = \Psi = \Psi' = 0 \quad \text{at} \quad \xi = \pm\frac{1}{2} \tag{8}$$

For completeness we have also considered the corresponding case of free boundaries characterized by

$$\Theta = \Psi = \Psi'' = 0 \quad \text{at} \quad \xi = \pm\frac{1}{2} \tag{9}$$

Differential Equations for Model B

The separated form for the disturbances is again assumed as in Equation (5). For the analysis of overstability it is convenient, though not necessary, to set $\tau_R = -\infty$, thus reducing Equation (2) to the model of Bird and Carreau (8). After evaluating the strain tensors, integrating to obtain the stress components and putting the result into Equations (4), the following differential equations are derived

$$\begin{aligned} &(D^2 - \epsilon^2 - N_{Pe}\sigma)\Theta = -\epsilon^2 N_{Ra}\Psi \\ &(1 + \sigma\lambda)^2\Theta = -N_{Re}\sigma(1 + \sigma\lambda)^2(D^2 - \epsilon^2)\Psi \\ &\quad + (1 + \sigma\lambda)(D^2 - \epsilon^2)^2\Psi + B\epsilon^2\left(1 + \frac{\sigma\lambda}{2}\right)(D^2 - \epsilon^2)X \\ &-\sigma N_{Re}(1 + \sigma\lambda)^2 X + (1 + \sigma\lambda)(D^2 - \epsilon^2)X \\ &\quad = A\left(1 + \frac{\sigma\lambda}{2}\right)(D^2 - \epsilon^2)\Psi - N_{Re}(1 + \sigma\lambda)^2\Psi \end{aligned} \tag{10}$$

where

$$A = \frac{\sum_p 2m_p \lambda^3_{2p}}{\sum_p m_p \lambda^2_{2p}} (\delta/2 + 1)\,\dot{\gamma}$$

$$B = \frac{\sum_p m_p \lambda^3_{2p}}{\sum_p m_p \lambda^2_{2p}}\,\delta\,\dot{\gamma}$$

$$m_p = \frac{\eta_p}{\lambda^2_{2p}(1 + 2\lambda^2_{1p}\,\mathrm{II})}$$

$$\lambda = \sum_p \lambda_{2p}$$

This gives a set of cubic equations for σ, the complex growth rate. If Θ, Ψ, and X were known the boundary conditions could be employed to generate an equation for σ. Unfortunately, this is not the case and only special cases of Equation (10) will be examined. If $\lambda = 0$, the resulting equations are those for a Newtonian fluid (note $\lambda = 0$ implies that all λ_{2p} are zero and therefore that $A = B = 0$). This case has been examined by previous authors. If the product $\sigma\lambda << 1$ we may write, neglecting terms of $O(\sigma\lambda)$ or smaller

$$\begin{aligned} &-\sigma N_{Re}X + (D^2 - \epsilon^2)X = A(D^2 - \epsilon^2)\Psi - N_{Re}\Psi \\ &N_{Re}\sigma(D^2 - \epsilon^2)\Psi + \Theta = (D^2 - \epsilon^2)^2\Psi + B\epsilon^2(D^2 - \epsilon^2)X \\ &-N_{Ra}\epsilon^2\Psi = (D^2 - \epsilon^2 - N_{Pe}\sigma)\Theta \end{aligned} \tag{11}$$

These equations should describe the stability criterion for a very slightly viscoelastic fluid. On the other hand, if λ is not small but $\sigma << 1$, Equations (10) may be written as

$$\begin{aligned} &-\sigma N_{Re}X + (1 + \sigma\lambda)(D^2 - \epsilon^2)X = A\left(1 + \frac{\sigma\lambda}{2}\right) \\ &\qquad (D^2 - \epsilon^2)\Psi - N_{Re}\Psi(1 + 2\sigma\lambda) \\ &N_{Re}\sigma(D^2 - \epsilon^2)\Psi + \Theta(1 + 2\sigma\lambda) = (1 + \sigma\lambda)(D^2 - \epsilon^2)\Psi \\ &\qquad + B\epsilon^2\left(1 + \frac{\sigma\lambda}{2}\right)(D^2 - \epsilon^2)X \\ &-N_{Ra}\epsilon^2\Psi = (D^2 - \epsilon^2 - N_{Pe}\sigma)\Theta \end{aligned} \tag{12}$$

where terms of $O(\sigma^2)$ or smaller have been dropped. These equations should be valid for moderately viscoelastic liquids very near to the neutral stability curve ($\sigma_r = 0$) if σ does not jump discontinuously to $\sigma_r >> 0$ for conditions slightly removed from neutral stability.

Equations (10), (11), and (12) are subject to the boundary conditions given in Equations (8) and (9) with the added condition

$$X = 0 \quad \text{at} \quad \xi = \pm\frac{1}{2} \tag{13}$$

If the exchange of stability assumption is valid Equation (10) reduces to those given in our previous paper (5) (allowing arbitrary τ_R). After eliminating X

$$\begin{aligned} \Theta &= (D^2 - \epsilon^2)^2\Psi + A\,B\epsilon^2(D^2 - \epsilon^2)\Psi - B\epsilon^2 N_{Re}\Psi \\ (D^2 - \epsilon^2)\Theta &= -\epsilon^2 N_{Ra}\Psi \end{aligned} \tag{14}$$

Solution of the Equations

Solution of the set of Equations (6), (11), and (12) was accomplished by means of an extension, due primarily to Finlayson (*16, 17*), of the Galerkin technique (*18, 19*). The dependent variables are expanded in terms of a set of trial functions

$$\begin{aligned} \Psi &\cong \sum_1^N a_r(t)\psi_n \\ X &\cong \sum_1^N b_r(t)\chi_n \\ \Theta &\cong \sum_1^N c_r(t)\theta_n \end{aligned} \tag{15}$$

where the ψ_n, χ_n, and θ_n form complete sets as $n \rightarrow \infty$. The difference from conventional applications is that the coefficients of the trial functions are allowed to be time dependent. The trial functions are chosen such that the boundary conditions are satisfied exactly. For example, for the case of fixed boundaries we use

$$\begin{aligned} \psi_n &= \xi^{n-1}(\xi^2 - 1/4)^2 \\ \chi_n &= \xi^{n-1}(\xi^2 - 1/4) \\ \theta_n &= \xi^{n-1}(\xi^2 - 1/4) \end{aligned} \quad (16)$$

Putting these trial solutions into a modified form of the differential equations and making the residual spatially orthogonal to the trial functions (*17*), one generates a set of 3N differential equations for the 3N unknown coefficients. The system of 3N equations can be represented in matrix form by

$$\underline{\underline{H}} \frac{d\underline{\underline{A}}}{dt} = \underline{\underline{B}}\,\underline{\underline{A}}$$

In all cases of interest we expect H to be nonsingular. Then

$$\frac{d\underline{\underline{A}}}{dt} = \underline{\underline{H}}^{-1}\underline{\underline{B}}\,\underline{\underline{A}} \equiv \underline{\underline{D}}\,\underline{\underline{A}} \quad (17)$$

The notation is analogous to that of Finlayson (*17*). A is a column matrix such that $\underline{A}^T = (a_1, a_2, \ldots a_N, b_1, b_2, \ldots b_N, c_1, c_2, \ldots c_N)$, and the elements H_{ij} and B_{ij} contain terms of the form $[\psi_i, (D^2 - \epsilon^2)\psi_j]$, $[\psi_i, (D^2 - \epsilon^2)\theta_j]$, etc., where

$$[u_i, v_j] = \int_{-1/2}^{1/2} u_i v_j d\xi$$

Finlayson has shown that if the time dependent coefficients $a_i(t)$, $b_i(t)$, $c_i(t)$ of the trial functions can be expressed by $a_i(t) = a_i e^{\sigma t}$, etc., where the $\overline{a_i}$ are constants, then the growth factor σ is an eigenvalue of the matrix $\underline{\underline{D}}$. Though conditions for neutral stability of the system can be found without actually calculating the eigenvalues of $\underline{\underline{D}}$ (*17*), we wish to be certain that permissible values of σ_r do not jump discontinuously at neutral stability from $\sigma_r << 0$ to $\sigma_r >> 0$. Such a situation, which has been noted in some stability problems (*20* to *22*), is not consistent with the rheological models employed here. To check this, eigenvalues of $\underline{\underline{D}}$ were calculated using a double QR iteration scheme on an IBM 360/91 (*23*).

The critical Rayleigh number is determined by searching the parameters to find where the first real part of one of the eigenvalues just becomes positive. If at this point the imaginary part is zero, then exchange of stabilities is valid for the set of parameters in question.

Case 1. Models A and B—Exchange of Stabilities

Differential Equations for Model B. For the case where $\sigma = 0$ it is important to note that Equations (7) and (14) have exactly the same form. Thus for the case where exchange of stabilities is valid we can proceed with the solution of Equations (7), knowing that the results are readily applied to Model B by merely redefining the parameters in accord with Equations (14) (5).

Use of Galerkin's method to obtain an approximate solution for both fixed and free boundaries has been described. Since Equations (7) and (14) are biquadratic, they may also be solved exactly (9). We use Equation (7) as an example. Eliminating Θ one obtains

$$(D^2 - \epsilon^2)^3\Psi + GF\epsilon^2(D^2 - \epsilon^2)^2\Psi - G\epsilon^2 N_{Re}(D^2 - \epsilon^2)\Psi = N_{Ra}\epsilon^2\Psi \quad (18)$$

The general solution of (18) can be expressed as a superposition of solutions of the form

$$\Psi = e^{\pm\Lambda\xi} \quad (19)$$

where Λ^2 is a root of the equation

$$(\Lambda^2 - \epsilon^2)^3 + GF\epsilon^2(\Lambda^2 - \epsilon^2)^2 - G\epsilon^2 N_{Re}(\Lambda^2 - \epsilon^2) + N_{Ra}\epsilon^2 = 0 \quad (20)$$

Equation (19) can be split into an even and odd solution

$$\begin{aligned} \Psi_E &= \overline{A}\cosh \Lambda_1\xi + \overline{C}\cosh \Lambda_2\xi + \overline{E}\cosh \Lambda_3\xi \\ \Psi_O &= \overline{B}\sinh \Lambda_1\xi + \overline{D}\sinh \Lambda_2\xi + \overline{F}\sinh \Lambda_3\xi \end{aligned} \quad (21)$$

Considering the even solutions for the fixed boundary conditions

$$\left.\begin{aligned} &\Psi = D\Psi = 0 \\ &(D^2 - \epsilon^2)^2\Psi + GF\epsilon^2(D^2 - \epsilon^2)\Psi - \epsilon^2 GN_{Re}\Psi = 0 \end{aligned}\right\} \text{at } \xi = \pm\frac{1}{2} \quad (22)$$

we require that the following determinant must vanish:

$$\begin{vmatrix} 1 & 1 & 1 \\ \Lambda_1 \tanh \frac{\Lambda_1}{2} & \Lambda_2 \tanh \frac{\Lambda_2}{2} & \Lambda_3 \tanh \frac{\Lambda_3}{2} \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0 \quad (23)$$

where $\beta_i = (\Lambda_i^4 - 2\epsilon^2\Lambda_i^2 + \epsilon^4) + FG\epsilon^2(\Lambda_i^2 - \epsilon^2) - \epsilon^2 GN_{Re}$.

The odd solutions yield the following determinant, which must also vanish at the neutrally stable condition

$$\begin{vmatrix} 1 & 1 & 1 \\ \Lambda_1 \coth \frac{\Lambda_1}{2} & \Lambda_2 \coth \frac{\Lambda_2}{2} & \Lambda_3 \coth \frac{\Lambda_3}{2} \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0 \quad (24)$$

By varying the physical parameters one can determine whether the even or odd mode of stability occurs and at what values of these parameters instability can occur.

For free boundaries the solutions must be of the form

$$\Psi_n = \overline{A}\sin n\pi(\xi + 1/2) \quad (25)$$

This gives the Rayleigh number directly

$$N_{Ra} = \frac{(n^2\pi^2 + \epsilon^2)^3 - FG\epsilon^2(n^2\pi^2 + \epsilon^2)^2 - GN_{Re}\epsilon^2(n^2\pi^2 + \epsilon^2)}{\epsilon^2} \quad (26)$$

For given physical parameters a search over ϵ gives the critical Rayleigh number. (This proved more efficient than setting $\partial N_{Ra}/\partial\epsilon = 0$ and solving for the critical wave number.)

Case 2. Model A

Only those cases where $\sigma_r = \sigma_i = 0$ were studied. Though by no means certain, one might expect that the primary importance of this mode of marginal stability would carry over from Case 1. Stress disturbances are readily found to be

$$S'_{xx} + p' = 2\alpha_0 v'_{x,x} + 2\alpha_1 \dot{\gamma}\, (v'_{x,y} + v'_{y,x}) + 2\alpha_2 \dot{\gamma}\, (2v'_{y,x} + y v'_{x,xx})$$

$$S'_{yy} + p' = 2\alpha_0 v'_{y,y} + 2\alpha_1 \dot{\gamma}\, (v'_{x,y} + v'_{y,x}) + 2\alpha_2 \dot{\gamma}\, (2v'_{x,y} + v'_{y,x} + y v'_{y,xy})$$

$$S'_{zz} + p' = 0$$

$$S'_{yx} = S'_{xy} = \alpha_0 (v'_{x,y} + v'_{y,x}) + 2\alpha_1 \dot{\gamma}\, (v'_{x,y} + v'_{y,y}) + \alpha_2 \dot{\gamma}\, (v'_{y,y} + v'_{x,x}) + y(v'_{y,xx} + v'_{x,xy}) \quad (27)$$

$$S'_{zx} = S'_{xz} = \alpha_0 v'_{z,x} + \alpha_1 v'_{z,y} + \alpha_2 \dot{\gamma}\, y v'_{z,xx}$$

$$S'_{yz} = S'_{zy} = \alpha_0 v'_{z,y} + \alpha_1 \dot{\gamma}\, v'_{z,x} + \alpha_2 \dot{\gamma}\, (v'_{x,x} + y v'_{x,xy})$$

As before, we assume that the disturbances can be expressed in separated form

$$\frac{v'_x}{U} = i\Psi_{,\xi}\, e^{i\epsilon\nu}$$

$$\frac{v'_y}{U} = \epsilon\Psi\, e^{i\epsilon\nu}$$

$$\frac{v'_z}{U} = -\,\epsilon X\, e^{i\epsilon\nu} \quad (28)$$

$$\frac{T'}{T_1 - T_2} = \epsilon T\, e^{i\epsilon\nu}$$

$$\frac{p'}{\rho_1 U^2} = \epsilon P\, e^{i\epsilon\nu}$$

where $i = \sqrt{-1}$, $\nu = x/d$, $\xi = y/d - 1/2$, and $\eta = z/d$. From the conservation equations one eventually obtains

$$(D^2 - \epsilon^2)\Theta - i\xi\epsilon N_{Re}\, N_{Pr}\Theta = -\, N_{Ra}\epsilon^2\Psi$$
$$(29)$$
$$(D^2 - \epsilon^2)^2\Psi - \Theta = i\xi N_{Re}\epsilon(D^2 - \epsilon^2)\Psi + iJ\epsilon[2(D^3 - \epsilon^2 D)\Psi - \xi(D^2 - \epsilon^2)^2\Psi]$$

where Θ, N_{Re}, N_{Ra}, N_{Pr}, and J are defined as before.

It will be noted that Equations (29) have complex coefficients. Another interesting feature is that α_1 does not appear in these equations. Thus, α_1 and, consequently, $(A_1')^2$ [see Equation (1)] do not affect the stability of the flow to disturbances spatially dependent only upon the x and y coordinates. The stability is only affected by normal stresses in the same coordinate directions as the disturbance dependencies ($\alpha_2 = S_{xx} - S_{yy}$). A somewhat similar result is found for Case 1, where $S_{yy} - S_{zz}$ plays the predominant role [Chun and Schwarz (*24*) for plane Poiseuille results].

For the case where $J = N_{Re} = 0$, Equations (29) reduce to the classical Newtonian Bénard problem

$$(D^2 - \epsilon^2)^3\Psi = -\, \epsilon^2 N_{Ra}\Psi \quad (30)$$

In this case, however, it is interesting to note that in the presence of flow ($N_{Re} \neq 0$) decoupling of the flow and heat transfer portions of the problem does not occur, even with Newtonian fluids. With $N_{Re} \neq 0$ and $J = 0$, Equations (29) reduce to

$$(D^2 - \epsilon^2)^2\Psi - \Theta = i\xi N_{Re}\epsilon(D^2 - \epsilon^2)\Psi$$
$$(31)$$
$$(D^2 - \epsilon^2)\Theta - i\xi\epsilon N_{Pe}\Theta = -\, \epsilon^2 N_{Ra}\Psi$$

This is the form given by Gallegher and Mercer (*13*) and Deardorff (*14*) for their analyses of the Newtonian fluid problem.

The boundary conditions for Equations (29) are

$$\left.\begin{array}{l}\Psi = \Psi' = 0\\ \Theta = 0\end{array}\right\} \text{at} \quad \xi = \pm\frac{1}{2} \text{ for fixed boundaries}$$

$$\left.\begin{array}{l}\Psi = \Psi'' = 0\\ \Theta = 0\end{array}\right\} \text{at} \quad \xi = \pm\frac{1}{2} \text{ for free surfaces}$$

Equations (29) were solved approximately using a Galerkin technique. However, in this case complex matrix manipulation was necessary and investigations were limited by time restrictions.

RESULTS AND DISCUSSION

On the Exchange of Stabilities

For the set of differential Equations (6), and (10) to (12) the principle of exchange of stabilities cannot be proved. The modified Galerkin technique was used to determine if oscillatory modes of instability were possible in the range of fluid and flow parameters of interest. The relevant physical parameters are the Prandtl number, a pseudo-Reynolds number (one should remember that the inertial terms in the basic flow vanish identically), two normal stress parameters, the Rayleigh number, the wave number of the disturbances, and the exponential time factor σ. This time factor arising in Equations (6), (11) and (12) was determined directly by the Galerkin method as the eigenvalues of the matrix $\underline{D}$. Material functions were evaluated by first choosing a shear rate $\dot{\gamma}$. Then, using data of Tanner (*25*), normal stress parameters for the chosen value of $\dot{\gamma}$ were estimated and Reynolds number, for a given flow geometry, was computed. Two- and three-term expansions in all three unknown functions [see Equations (15)] were solved numerically by picking a series of Rayleigh numbers and finding the corresponding value of wave number ϵ which gave a zero value to one of the real parts of the eigenvalues of $\underline{D}$ (that is, the real part of σ). If the imaginary part of this eigenvalue is zero, then the exchange of stabilities hypothesis is valid for the set of parameters. The difference in critical Rayleigh number between two- and three-term expansions was always less than 3%. The three-term expansion agrees with the known Newtonian results to three significant figures. Because of the computation required, extremes of physically meaningful values for flow and material parameters were investigated first, followed by spotchecks at intermediate values. In no case was an oscillatory instability found from Equation (6) over the range of $-50 \leq N_{Re}G \leq 10$, $-1 \leq J \leq 0$, $10 \leq N_{Pr} \leq 500$, and $-5 \leq FG \leq 1$. Equations (11) and (12) were examined over the parameter ranges $0 \leq N_{Re} \leq 50$, $10 \leq N_{Pr} \leq 500$, $0 \leq A \leq 2$, $-2 \leq B \leq 1$, and $0 \leq \lambda \leq 100$. No oscillatory instabilities were found. Thus one observes from comparison of Equations (6) with (7) and Equations (10) to (12) with (14) that the only pertinent variables are N_{Re}, F, and G for Model A, and N_{Re}, A, and B for Model B. Some results are shown in Figure 2 over the range of parameters studied for both Model A and Model B at low wave number. Critical Rayleigh numbers and critical wave numbers for the two models are indistinguishable. However, for Model A at any given N_{Re}, there is a wave number above which the flow is unstable for all Rayleigh numbers (Figure 3). Near the transition wave number, however, σ is very large and discontinuous. Similar physically unrealistic time-dependent behavior of second-order fluid models has been observed elsewhere (*20, 27*). For Model B, however, no such high frequency

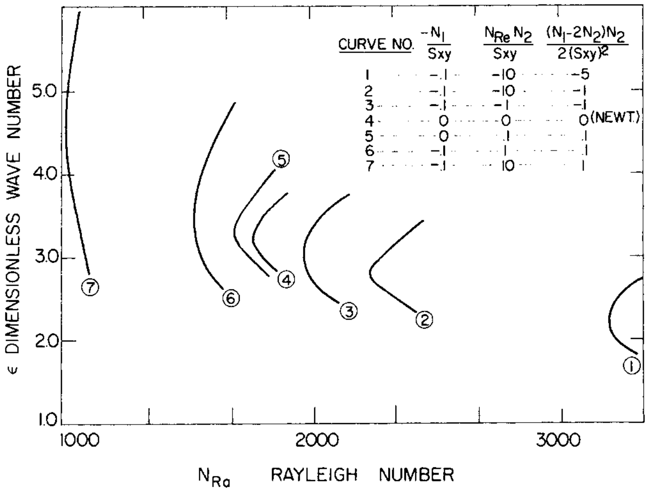

Fig. 2. Neutral stability curves, including the possibility of oscillatory instabilities.

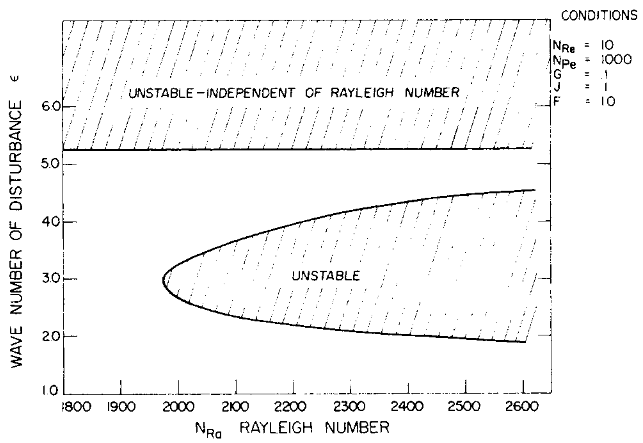

Fig. 3. Example of stability analysis with Model A. Results at high wave number.

instability is possible, and the eigenvalues (σ) are well behaved at all wave numbers. This result suggests that the second-order fluid analysis may give at least qualitative information about hydrodynamic stability problems of viscoelastic fluids if σ is very small near the neutral stability curve. If the second-order fluid analysis leads to the prediction of an instability with very large and discontinuous σ near the neutral stability line, a more realistic model must be used, since the theoretical foundations for the second-order fluid are not compatible with large values of σ (*20*). Thus it is important to know the actual values of the eigenvalues as well as their sign when working with restrictive fluid models.

Disturbance Case 1

Since the validity of exchange of stabilities was shown for both fluid models over the fluid parameter range of interest, detailed investigations were made using the simpler set of differential equations (7) and (14). Here Model A and Model B have exactly the same form of differential equations. Both the exact solution and the results of the Galerkin approximation technique were used, and it was found that three terms in the Galerkin expansions were sufficient to obtain agreement in the critical Rayleigh number to within 1%.

The results for the free-free and fixed-fixed boundaries are shown in Figures 4 and 5, respectively. The second normal stress function is seen to be extremely important in determining the stability of the system. Negative second normal stress differences tend to stabilize the system relative to the Newtonian result, while positive second normal stress tends to destabilize the system. In an earlier paper the effect of N_1 was essentially eliminated by studying, in effect, slightly elastic fluids at large Reynolds number. In this paper the effect of N_1 is shown. It is expected to be important for applications at low Reynolds number and high shear rate. An example could be the manufacture of film from viscoelastic fluids.

For certain values of the normal stress parameters ($GN_{Re} > 68$, $FG > 1.1$, see Table 1 of reference *5* for connection with normal stresses) an elastic instability which requires no buoyancy effect (that is, one can have $N_{Ra} = 0$) is generated. This was predicted and discussed by Giesekus (*3*) and has no analogy in Newtonian fluids.

In all cases studied the critical wave number increases as the stability of the system decreases and decreases as the stability of the system increases. This is shown in Figure 6.

Disturbance Case 2

We have seen that the differential equations for the disturbances considered to be a function of the flow (x) and y coordinates involve complex coefficients and depend on only one of the elastic parameters, α_2. Gallegher and Mercer (*13*) and Deardorff (*14*), examining the Newtonian case, have found that for the set of Equations (31) both the flow (N_{Re}) and the Prandtl number are stabilizing. Thus, the most unstable case is a disturbance in the y, z plane, which is examined above. Turning to the non-

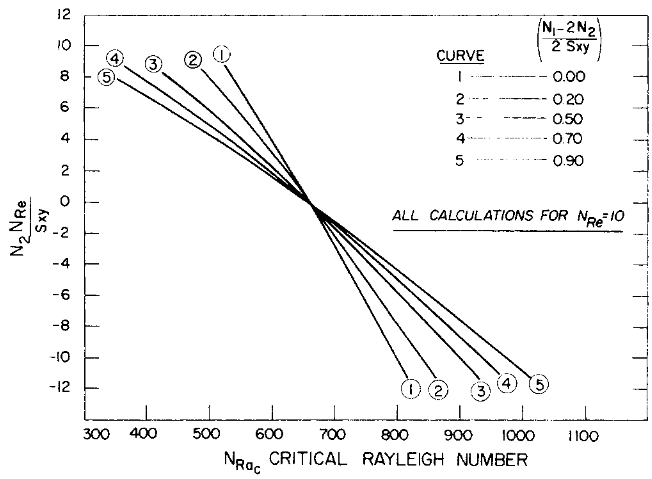

Fig. 4. Effect of normal stress parameter on the critical Rayleigh number. Free-free boundaries.

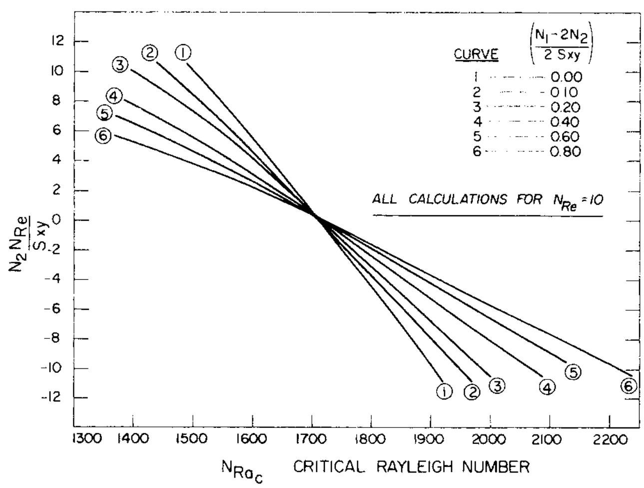

Fig. 5. Effect of normal stress parameter on the critical Rayleigh number. Fixed-fixed boundaries.

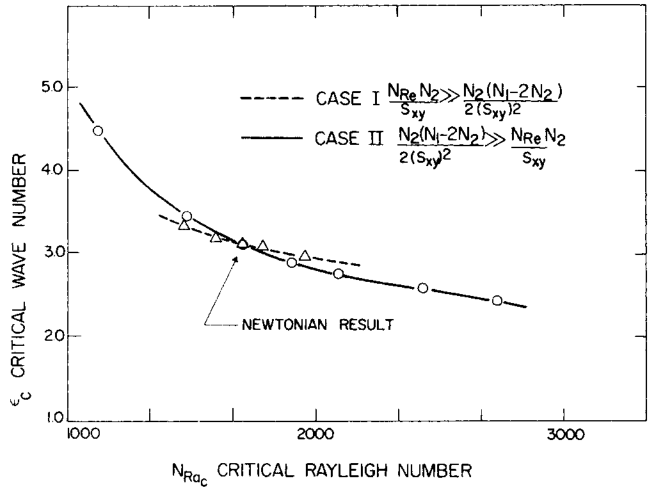

Fig. 6. Relation between critical wave number and critical Rayleigh number.

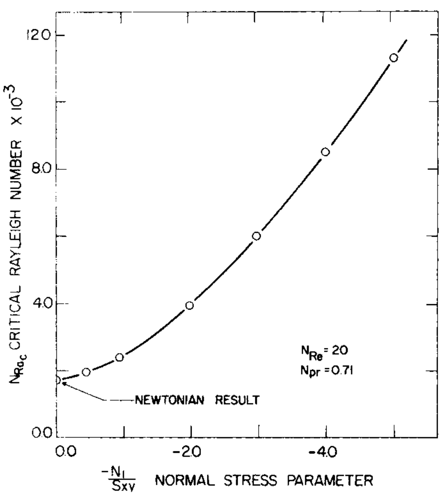

Fig. 7. Effect of normal stress parameter ($-$ N_1/S_{xy}) on the critical Rayleigh number for disturbance form 2.

Newtonian case, it is not obvious that, for Model A, the additional non-Newtonian term which arises in the second of Equations (29) for $J \neq 0$ is also stabilizing. There seems to be no controversy over the sign of this term, since in simple shearing flow $J = -N_1/(2S_{xy})$ (see Table 1 of reference 5), so only nonpositive values of J were studied. The range of elastic parameter investigated numerically was $-5 \leqq (-N_1/S_{xy}) \leqq 0$. The numerical results indicate that elasticity is also stabilizing in the range of parameters investigated. This result is shown in Figure 7.

It is concluded that (y, z)-dependent disturbances are less stable than (x, y)-dependent disturbances. Thus, the former will determine the critical Rayleigh number for the range of physical parameters studied.

Uses and Limitations

The use of Model A in hydrodynamic stability analysis requires careful consideration. Pipkin and Owen (*26*) have shown that 13 material functionals may be necessary to describe perturbations of an incompressible simple fluid in viscometric flow. If the variation in material functions (α_i) is assumed negligible in the stability analysis, one is essentially using a specific second order fluid. Coleman, Duffin, and Mizel (*27*) have shown the potential hazards in using this model in certain flow domains. For the "generalized" Model A to be a valid model for this analysis the terms generated from perturbations in α_i must be negligible (*5*), and the growth rate of perturbations near the neutral curve must be small, that is, $\sigma_r << 1$ (*20*).

Some indication of the validity of the former assumption was sought by performing a few computations in which variation of α_0 due to the perturbation in strain rate was considered. This was done by using a power-law expression for α_0 and retaining the term α_3 in reference 5, which is a measure of the effect of the flow perturbation on α_0. Variations in α_1 and α_2 were not considered. In all cases computed, which were for $N_{Re} = 10$ and several values of normal stress differences, the value of N_{Ra_c} changed by 5% or less for power-law exponents between 0.6 and 1.0.

Model B is thought to be more widely applicable than Model A, and it seems to have some predictive value for certain non-viscometric flows (*6, 28, 29*). That it reduces to the same form of differential equation as Model A is encouraging. As suggested by Ginn and Denn (*1*) if this "viscometric hypothesis" is correct, stability analysis may be a valuable means of determining rheological properties.

CONCLUSIONS

1. Over the range of parameters studied, oscillatory instabilities do not occur for Model A or Model B in plane Couette flow with a superposed temperature gradient.
2. Disturbances which are dependent on the coordinates perpendicular to the flow direction (Case 1) are less stable (lower N_{Ra_c} at the same values of other parameters) than disturbances dependent upon the flow and vertical directions (Case 2). This was true over the entire range of material and flow parameters studied.
3. The approximate Galerkin method gives excellent agreement with the exact solution when comparison is possible. The Galerkin technique would seem to be a good method of attack on more difficult stability problems where eigenvalues of the problem are of more interest than the exact form of the solution to a differential equation.
4. Use of second-order fluids in the analysis of hydrodynamic stability problems should be restricted to cases where the growth rate of the disturbance is very small near the neutral stability curve.

ACKNOWLEDGMENT

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NOTATION

$$A = \frac{\sum_p 2m_p \lambda^3_{2p}}{\sum_p mp\, \lambda^2_{2p}} (\delta/2 + 1)\, \dot{\gamma}$$

$\underline{\underline{A}}$ = matrix used in Equation (17)

$\mathbf{A}_1$ = first Rivlin-Ericksen tensor = 2D

$\mathbf{A}_{n+1} = D\mathbf{A}_n/Dt + \mathbf{A}_n \cdot \mathbf{\Omega} - \mathbf{\Omega} \cdot \mathbf{A}_n + \mathbf{D} \cdot \mathbf{A}_n + \mathbf{A}_n \cdot \mathbf{D}$ $(n \geqq 1)$

$\mathbf{B}$ = strain measure which has Cartesian components $B_{ij} = (\partial x_i/\partial x'_\alpha)(\partial x_j/\partial x'_\alpha) - \delta_{ij}$

$\underline{\underline{B}}$ = matrix used in Equation (17)

$$B = \frac{\sum_p mp\, \lambda^3_{2p}}{\sum_p m_p \lambda^2_{2p}} \delta\, \dot{\gamma}$$

$C = \sum_p m_p \lambda^2_{2p}$

C = strain measure which has Cartesian components $C_{ij} = (\partial x'_\alpha/\partial x_i)(\partial x'_\alpha/\partial x_j) - \delta_{ij}$

$D = d/d\xi$

D $= 1/2\,[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$

$\underline{\underline{D}} = \underline{\underline{H}}\ \underline{\underline{B}}$

d = distance between plates

$F = (\alpha_1 + \alpha_2)\,\dot{\gamma}/\alpha_0$

$G = (\alpha_1 + 2\alpha_2)\,\dot{\gamma}/\alpha_0$

$g = |\mathbf{g}|$

g = acceleration due to gravity

$\underline{\underline{H}}$ = matrix used in Equation (17)

I = unit tensor

$J = \alpha_2\,\dot{\gamma}/\alpha_0$

$$m[(t-t'),\ \mathrm{II}(t')] = \sum_p \frac{\eta_p}{\lambda^2_{2p}} \frac{\exp[-(t-t')/\lambda_{2p}]}{[1 + 2\mathrm{II}(t')\lambda^2_{1p}]}$$

$m_p = \eta_p/[\lambda^2_{2p}(1 + 2\lambda^2_{1p}\,\mathrm{II})]$

$N_1 = S_{xx} - S_{yy}$ (see Figure 1)

$N_2 = S_{yy} - S_{zz}$ (see Figure 1)

$N_{Pe} = N_{Pr}N_{Re}$

$N_{Pr} = \alpha_0/(\rho_1\kappa)$ for Model A; $= C/(\rho_1\kappa)$ for Model B

N_{Ra} = Rayleigh number $= \rho_1 g\alpha\beta d^4/(\kappa\alpha_0)$ for Model A; $= (\rho_1 g\alpha\beta d^4)/(\kappa C)$ for Model B

N_{Re} = Reynolds number $= \rho_1 d^2\,\dot{\gamma}/\alpha_0$ for Model A; $= \rho_1 d^2\,\dot{\gamma}/C$ for Model B

P = pressure perturbation, defined in Equation (5)

p = pressure

S = stress tensor

T = temperature, including $T(\xi)$ the ξ-dependent portion of the temperature perturbation defined in Equation (5)

t = time

$\bar{t} = (U/d)t$

$U = \dot{\gamma}\,d$

V = specific volume

$v = |\mathbf{v}|$

v = velocity

II = second invariant of $\mathbf{D} = D_{ij}\,D_{ji}$

Greek Letters

$\alpha = (1/V)(\partial V/\partial T)$

α_i = coefficients, taken to be functions of strain rate, defined in Equation (1)

$\beta = (T_1 - T_2)/d$

β_i = defined, following Equation (23)

$\dot{\gamma}$ = shear rate in laminar shear flow $v_i = (\dot{\gamma}\,y, 0, 0)$

δ = constant in Equation (2) which, for laminar shearing flow, is a measure of N_2/N_1

δ_{ij} = Kronecker delta ($\delta_{ij} = 1$ for $i = j$, $= 0$ for $i \neq j$)

ϵ = dimensionless wave number, defined in Equation (5)

$\eta = z/d$

η_p = material constant for Model B

$\Theta = \rho_1\alpha d^2 g(T_1 - T_2)\epsilon^2 T(\xi)/(U\alpha_0)$

θ_n = trial function defined in Equation (15)

κ = thermal diffusivity

Λ = root of Equation (20)

Λ_i = a particular root to Equation (20)

$\lambda = \sum_p \lambda_{2p}$, made dimensionless with U/d when necessary

$\lambda_{1p}, \lambda_{2p}$ = material constants

$\nu = x/d$

$\xi = y/d - 1/2$

ρ = density

σ = complex disturbance growth rate $= \sigma_r + i\sigma_i$, made dimensionless with U/d when necessary

σ_r = real part of disturbance growth rate

σ_i = imaginary part of disturbance growth rate

$\boldsymbol{\tau} = \mathbf{S} + p\mathbf{I}$

τ_R = rupture time [Equation (2)]

X = portion of velocity perturbation, defined by Equation (5)

χ_n = trial function defined in Equation (15)

Ψ = portion of velocity perturbation, defined in Equation (5)

ψ_n = trial function defined in Equation (15)

$\boldsymbol{\Omega} = 1/2\,[(\nabla \mathbf{v})^T - (\nabla \mathbf{v})]$

Superscript

$'$ = dummy variable when applied to t; otherwise indicates a perturbation from the value of the unsuperscripted quantity in undisturbed flow

Subscripts

c = state of critical stability

m = state of marginal stability

1 = evaluated at lower plate

2 = evaluated at upper plate

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